UNCLASSIFIED

AD NUMBER

AD483015

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; 1964. Other requests shall be referred to Naval Postgraduate School, Monterrey, CA.

AUTHORITY

USNPS ltr, 28 Jun 1971

THIS PAGE IS UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

A CRITICAL SURVEY OF PERT/COST, WITH EMPHASIS ON THE MONTE CARLO TECHNIQUE OF NETWORK CALCULATION

by

John L. Underwood

Commander, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
OPERATIONS RESEARCH

United States Naval Postgraduate School
Monterey, California

1 9 6 4

4

A CRITICAL SURVEY OF PERT/COST, WITH EMPHASIS ON THE MONTE CARLO TECHNIQUE OF NETWORK CALCULATION

* * * * *

John L. Underwood

A CRITICAL SURVEY OF PERT/COST, WITH EMPHASIS ON
THE MONTE CARLO TECHNIQUE OF NETWORK CALCULATION

by

John L. Underwood

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
IN
OPERATIONS RESEARCH
from the
United States Naval Postgraduate School

Faculty Advisor

T. E. Oberbeck
Chairman
Department of Operations Research

Approved:

Academic Dean

ABSTRACT

The PERT/COST system, now in current use, requires several simplifying assumptions in order to permit the use of analytic methods for determining predictions of critical path activities and project duration. As a result of these assumptions, the predictions may involve significant error, and will always be optimistic. The Monte Carlo technique of network calculation does not require these assumptions and, hence, is capable of yielding more accurate predictions and providing more useful information. This technique is discussed in detail. A more flexible probability density model for activity times is introduced, and a resource allocation technique based on the probability that an activity will be on the critical path is developed. Finally, an application of PERT network theory to military operational planning is described.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

NOTATION

Symbol	Introduced on page	Meaning
$\_^*$	9	Indicates a random variable
t^*	9	Time required to complete an activity not yet completed
t	7	A specific value of the random variable t^*
t_e	11	Expected value, or mean, of t^*
t_s	25	Time allowed by schedule for completion of an activity
T^*	12	Elapsed time till the occurrence of an event, or until completion of an activity
T_E	12	Expected value, or mean, of T^*
T_S	26	Elapsed time at which an event is scheduled to occur
T_l^*	13	Latest elapsed time an event can occur and not delay the scheduled completion of the project, or the latest time an activity can be completed and not delay the project
T_L	13	Expected value, or mean, of T_l^*
S_E	25	Earliest elapsed time an activity/event can be scheduled for completion
S_L	25	Latest elapsed time at which an activity/event can be scheduled without causing a change in t_s for some subsequent activity
K^*	13	Slack for an activity/event, $= T_l^* - T^*$
K_E	14	Expected value, or mean, of K^*
$\sigma(\ *)$	10	Standard deviation of the random variable in parentheses
$\sigma^2(\ *)$	12	Variance of the random variable in parentheses
D^*	14	Duration of the entire project

NOTATION (cont'd)

Symbol	Introduced on page	Meaning
D_s	8	Elapsed time to the Directed Completion Date for the project
P_C	17	Probability an activity/event will be on the critical path. The Criticality Index.
P_S	14	Probability an activity/event will be completed on schedule
P_D	14	Probability that the project will be completed on or before the DCD
DCD	8	Directed Completion Date for the entire project
O	10	Optimistic estimate for t*, the smallest amount of time the activity can reasonably be expected to consume
M	10	Most likely estimate for t*. The elapsed time at which the activity is most likely to occur.
P	10	Pessimistic estimate for t*. The greatest amount of time the activity can be expected to consume.
x*	41	The standard Beta variable
[illegible]	41	The mode of x*

I. Introduction.

The PERT system was devised by a study group under the direction of the Special Projects Office, Bureau of Naval Weapons, Navy Department, in order to develop a methodology for providing the management of the Fleet Ballistic Missile (Polaris) program with an information reduction system for program monitoring and evaluation. By this system, management was to be continuously appriced of progress to date for the program as a whole, and be furnished valid predictions as to outlook toward accomplishing program objectives. [2]

The system, as developed, was a giant stride in management technology, and is credited with a major contribution toward the rather phenomenal success of the FBM development program. Not only did the PERT system accomplish the objectives already mentioned, but it allowed management to predict those activities in the development project whose completion time would have a direct effect on the overall project duration. These critical activities could then receive appropriate managerial attention.

The PERT system has since been employed extensively in industry, and has been made a standard procedure for monitoring research and development projects under the cognizance of the Federal government. Industry acceptance of the system has been, in general, enthusiastic and widespread.

The system has been recently extended to include a cost control feature, PERT/COST, which has now been designated the standardized system for monitoring federally sponsored R&D projects.

Another system, the Critical Path Method, has paralleled PERT in

development. It is similar in many respects to PERT, lacking PERT's stochastic representation of time, but incorporating a correlation between time and cost, enabling management to schedule optimally. The Department of Defense and NASA have incorporated the desirable features of both systems into PERT/COST.

The success of PERT and Pert-type systems has stimulated the interest of professionals in the fields of mathematics, management science, and operations research. The professional journals have featured many papers on the relative merits of the systems, proposals for refinement and extension of the systems, consolidation and integration of the systems, and the like. It is the purpose of this paper to consolidate many of these ideas, and to propose some refinements and extensions to the system, designed to extend its usefulness and increase the accuracy of predictions about the projects, and to generate specific recommendations for control of time and cost.

We shall first briefly describe the basic PERT system, critically discuss certain features of the system, and tender proposals by which the basic system may be improved. In particular, we shall discuss the Monte Carlo method of calculation of the PERT network, and show how this method overcomes many of the deficiencies inherent in the analytic method.

Next we shall describe the standard PERT/COST system currently being implemented by DOD and NASA. We shall follow this with our proposal for an integrated project control, scheduling, and resource allocation technique.

Finally, we shall discuss an extension of the PERT network approach to the problem of planning projects involving major future decisions, such as military campaigns.

2. Conclusions and Recommendations

As a result of this study, it is concluded that:

(a) The assumptions employed in the currently standard analytic PERT network calculations may induce significant error in the results.

(b) The standard analytic technique for network calculation fails to provide a sufficiently definitive measure for activity criticality.

(c) A more flexible model for activity completion time, providing a range of variance to allow for varying uncertainty in time estimates, is needed.

(d) The Monte Carlo technique does not depend on the assumptions which induce the inaccuracy in the analytic technique, hence is capable of yielding results limited in accuracy only by the validity of the estimates for activity time. In addition, this method yields the probability that a given activity will be on the critical path, and other useful information not obtained by the analytic technique. The Monte Carlo technique can easily provide the flexibility discussed in (c), above.

(e) The Monte Carlo technique requires increased computer running time, but the increase is not prohibitive, and is well justified by the improvement in quality of the results obtained.

(f) A resource allocation technique, based on the probability that an activity is on the critical path, is feasible, and provides a reasonable method of optimum resource allocation.

(g) The network representation of the complex interrelationships between activities in a project is a technique which may well be applied to the task of planning military operations.

It is recommended that the Monte Carlo technique for PERT network calculations be employed in lieu of the analytic technique. A method of utilizing this technique is discussed in detail in Appendix I.

3. The Basic PERT System

The basic PERT system is a means of defining the relationship between the various activities comprising a project, and for estimating the time required to complete each activity and the project as a whole. Use of the system enables realistic schedules to be imposed on the project, and assists in controlling the execution of the project.

By a project we shall mean a collection of tasks, or activities, each a necessary step toward the achievement of some final objective. Subsets, called paths, of these activities, are dependent sequentially, that is, any activity in the path, other than the beginning activity, may not be commenced until its predecessor has been completed. Each activity in the path, except the beginning activity, has a predecessor in the path. An example of a project in this sense may be as simple as the construction of a house, where the activities are tasks such as laying foundation, erecting subfloor, installing rough plumbing, erecting framing, installing rough wiring, etc. Or a project may be as complex as development of the Fleet Ballistic Missile system, with many hundreds of activities involved.

A network of directed arcs is a very convenient way of representing a project. The activities are represented by directed line segments, called arcs, terminating at nodes, called events. (See Figure 1.) The time required to complete an activity is associated with the length of the arc representing it. Events represent points in time. Several activities may terminate at one event. If so, the event is said to occur when the last of these terminating activities is completed. At that point in time, other activities, whose commencement is contingent upon

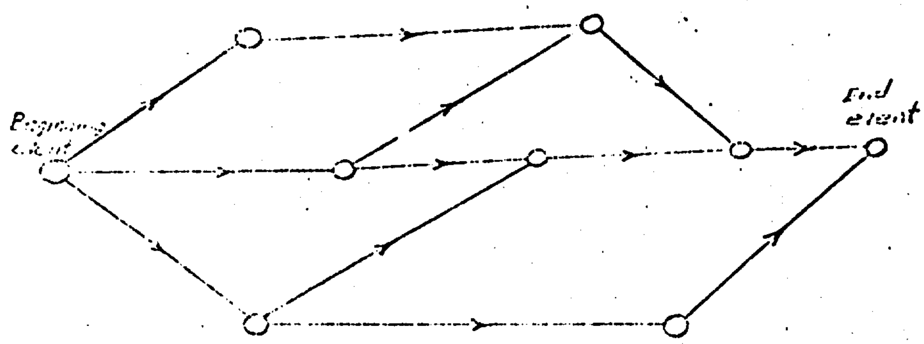


Figure 1

A Network Representation of a Project

the event may commence. These activities are represented by arcs directed [illegible] from the event. In the house construction example, the installation of the interior wallboard may commence only after the rough plumbing and rough wiring are completed.

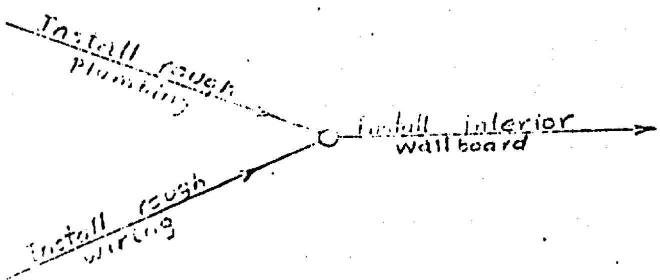


Figure 2

Two Independent Activities Incident to an Event

A project can always be represented by a network with one beginning [illegible] end event. Fictitious, or dummy, activities may be drawn

in the network in order to achieve this format. Such dummy activities may consume zero time, or may represent a predetermined amount of scheduled "dead" time. It is common practice for the network to be drawn with beginning event on the left, and with the end event, representing the achievement of the final objective, at the extreme right of the network.

A path may be defined as an unbroken chain of events, with origin at the beginning event and terminus at the end event. The network consists of many such paths, overlapping, paralleling, and crossing each other.

Activities generally consume time. The exact amount of time, t, which will be consumed in completing any one activity is not generally known in advance. If one looked at the history of a completed project, one could discover the exact amount of time, t, required to complete any given activity, and could label each event with its actual time of occurrence. The longest path, with respect to time, from the beginning event to the end event could then be found. This path is called the critical path. The activities on this path are called critical activities. The length of the critical path, the sum, $\sum t$, of the completion times for all activities on the path, is the total elapsed time from the beginning to the end of the project, and is called the duration, D. It is apparent that a small change in t for one of the activities on the critical path causes the same amount of change in D, but a small change in t for an activity not on the critical path would cause no change in D. Hence it is important that project management be able to predict the critical path in order to predict D, and in order to allocate resources, and fix

schedules, most effectively.

Any path, other than the critical path, is shorter than the critical path. The difference in lengths of the critical path and any other path is called the slack, K, for the non-critical path. An activity or event may be located on several paths simultaneously. The smallest slack associated with any path which includes a given activity or event is the value of slack associated with that activity/event. The activities/events on the critical path have zero slack. Any single activity may be delayed an amount less than or equal to its slack without affecting the duration of the project.

The term, slack, may have a slightly different meaning if some authority has imposed a Directed Completion Date, DCD, on the end event of the project. If so, we define the Scheduled Duration, D_s, as the elapsed time from the beginning event to the DCD. Then slack for the critical path is $(D_s - D)$. Slack for any other path is (D_s - path length). With this definition, slack for any activity may be either positive, negative, or zero, but all activities on the critical path will have the same [illegible] which will be less than that for any other activity in the network.

The [illegible] system is a method by which predictions of the critical [illegible] slack for any activity, may be made with sufficient accuracy [illegible] management to operate more effectively.

[illegible] Stochastic Network Model. In the planning stage, before a proj[illegible]d, the actual time, t, which will be required to complete [illegible]tivity, is an unknown quantity. It is unlikely that this [illegible] could be predicted exactly. We shall call such uncertain quantities

random variables, and symbolize them with an asterisk, as t^*. The corresponding unstarred symbol will represent a specific value of the variable. It is possible to set limits, within which t^* is almost certain to fall. It is also possible to estimate the shape of the probability density function governing the random variable, t^*. The probability density function for t^* is a curve whose ordinate in a region is a measure of the likelihood that t^* will fall in that region. To be more precise, the area under the curve in a region is the probability that t^* will occur in that region. In Figure 3 the ratio of the shaded area to the total area under the curve is the probability that t^* will fall between t_1 and t_2.

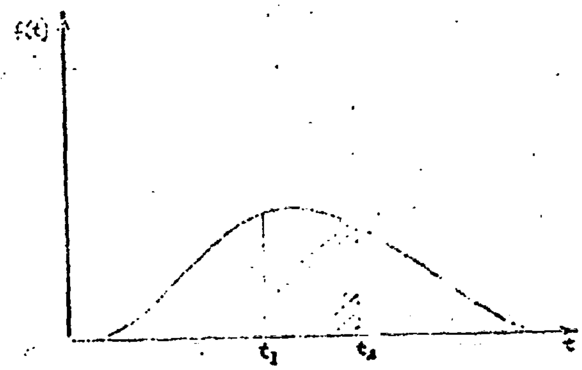


Figure 3

A Probability Density Function

The actual probability density function for t^* is difficult, if not impossible to obtain, and will be unique for any given activity performed in a given time period. Theoretically, if the activity were performed repeatedly, under the same conditions, with no learning taking place,

then a histogram showing the number of completions occurring in a time, $t \pm \Delta t$, plotted against t in discrete segments, would approximate the probability density function. Since this procedure is not possible, we must be satisfied with the best estimate we can formulate, using available knowledge of the nature of the activity.

In order to estimate the probability density function of t* for a given activity, the most qualified supervisor in charge of the activity makes three estimates of the completion time for the activity. One estimate, Ø, an <u>optimistic</u> estimate, is the <u>smallest</u> amount of time the activity may be reasonably expected to consume. Another, M, is the amount of time the activity is <u>most likely</u> to require. Finally, he makes a <u>pessimistic</u> estimate, P, which is the greatest amount of time the activity can be expected to consume, barring completely unforeseen [illegible]s. P and Ø form the upper and lower bounds, respectively, of the distribution of t*, and M is the mode of the density function. The [illegible] density function is assumed to have the value zero at P and Ø [illegible] reach a single peak at M.

[illegible] several well known mathematical functions exhibit the properties we have just assumed for the distribution of t*. The Beta function,

$$f(t) = k(t-a)^{\alpha} (b-t)^{\gamma} \tag{1}$$

with parameters a, b, α, and γ, was chosen as the model for the PERT system activity time because it could be specified to fit the constraints [illegible] estimates. The three estimates Ø, M, and P, specified three of [illegible] of the Beta function. The remaining parameter was specified [illegible] arbitrarily, requiring the standard deviation, $\sigma(t^*)$, to equal

$\frac{P-Ø}{6}$. One-sixth of the range is a frequently used estimator for the standard deviation of unimodal frequency distributions, hence was considered to be a reasonable assumption. For example, the standard normal distribution, truncated at ± 2.66, has its standard deviation equal to 1/6 the range. [6] As a result of the above requirements, one of the parameters, say α, becomes a function of the relative position of M in the range between Ø and P. Determination of this parameter requires the solution of a cubic equation, (see Appendix I). The exact determination of the mean, or expected value of t^*, symbolized by t_e, is a tedious process, since t_e is a function of a, b, α and γ. The following reasonably close linear approximation for the mean was adopted as a standard:

$$E[t^*] \cong t_e = \frac{Ø + 4M + P}{6} \qquad (2)$$

The probability density function thus assumed has a shape similar to that shown in Figure 4. This distribution will be referred to subsequently as the PERT Beta distribution.

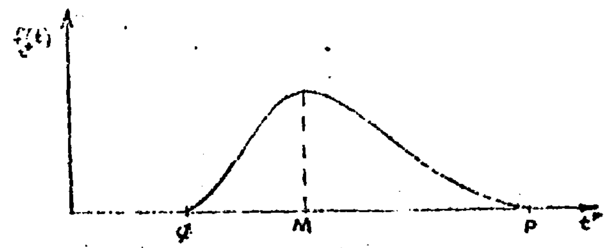


Figure 4

A Typical PERT Beta Density Function

A more detailed discussion of the PERT Beta distribution is included in Appendix I.

For purposes of predicting the critical path, the three estimates are combined by means of the weighted average formula, (2), for the mean. The expected value, or mean, t_e, is used as a deterministic value of t^* in the prediction of the critical path. t_e lies between M and the mid-point of (O,P), and is separated from M by 1/3 the distance between M and the midpoint. In this manner, account is taken of the usual tendency for estimators to be optimistic about placing M. The fundamental reason, however, for using t_e instead of M as an estimator for t^* lies in probability theory, rather than an effort to correct for this supposed bias.

In order to determine Slack, two new random variables, T^* and T_L^*, are defined. T^* is associated with a given event, and is the elapsed [illegible] the beginning event and the occurrence of the given event. T^* is the sum of the random variables, t^*, for the activities on the [illegible] path to the event. At this point in the development of the basic PERT system an assumption is made that the longest path to an [illegible] the path whose mean length is greatest. The mean length of [illegible] is the sum of the means, t_e, of the activities on the path. Then, [illegible] assumption, the mean, T_E, of T^*, is the largest of the mean lengths of all paths to the event. Operating on the basis of the same assumption, the further assumption is made that T^* is approximately normally distributed. This assumption is justified since we assume we know the longest path to the event, and since this path is the sum of n random variables, the Central Limit Theorem of probability theory asserts its [illegible] toward normality as n increases. Due to the assumed independence [illegible] activities along this path, the variance, $\sigma^2(T^*)$, of T^* is [illegible] the variances, $\sigma^2(t^*)$, of the activities on the path to the

event. The distribution of T^* is now completely specified by T_E and $\sigma^2(T^*)$ in view of the normality of T^*.

We may associate a random variable, T^*, with an activity as well as an event. We define T^* for an activity as the elapsed time until the activity is completed. Note that T^* for an activity is not necessarily equal to T^* for the event following the activity. T^* for that event is equal to the maximum of the T^*'s for the activities incident to it, that is, terminating at the event.

The random variable, T_L^*, for an event is defined as the latest time the event can occur and not delay the scheduled completion of the project. T_L^* is calculated by subtracting from D_s the sum of the t^*'s for all the activities on the longest path from the event to the end event. Again the assumption is made that this longest path will be the path whose mean length, $\sum t_e$, is longest. Then the mean value of T_L^*, written $T_{L*} = (D_s - \sum t_e)$, where the sum is the maximum of all such sums of paths between the given event and the end event. The variance of T_L^*, $\sigma^2(T_L^*)$, $= \sigma^2(t^*)$, along the path with greatest mean length.

A random variable, T_L^*, may also be associated with an activity. T_L^* for an activity is defined as the latest elapsed time since the beginning event at which the activity must be completed in order not to delay the scheduled completion of the project. Notice that T_L^* for any activity is the same as the T_L^* for the event following the activity.

It is apparent that the sum of the two paths used to determine T^* and T_L^* for a given event/activity is the longest path from the beginning to the end which contains that event/activity. From our definition of slack, we obtain the following relationship for slack, K^*:

$$K^* = D_s - \left[T^* \quad (D_s - T_L^*)\right] = T_L^* - T^* \qquad (3)$$

then the mean value, K_E, of slack is the difference, $T_L - T_E$. Notice that this expression for K_E depends upon the assumptions made in determining T_L and T_E.

The Probability of Meeting a Scheduled Completion Date. We have developed the random variable, T^*, whose distribution we have specified, by means of simplifying assumptions, to be normally distributed, with mean T_E and variance, $\sigma^2(T^*)$. The probability, P_S, that an event will occur before some scheduled completion date is simply $P[T^* < T_S]$, where T_S is the elapsed time from the beginning event to the scheduled completion date. In particular, the probability, P_D, that the project will be completed before the directed date is $P[D < D_s]$. Where the duration, D, is a random variable equal to T^* for the end event. These probabilities can be obtained from a table of values for the normal random variable.

The Predicted Critical Path is that path whose mean value of slack, is the least. All events or activities on this path will have the same value for K_E, which will be less than the K_E for all other events/activities.

This completes the description of the basic PERT system which is currently in general use. In practice, the network representing the project is drawn, events and activities are assigned identification numbers, and are assigned to specific organizations for responsibility. The contractors responsible for completion of each activity then submit their estimates, O, M, and P. Several computer programs have been written which take the activity number, preceding and succeeding event

numbers, and the three time estimates as inputs, then perform the network calculations. [5,7,15,20] The output lists the activities in any predetermined ordering, giving T_E, T_L, K_E the probability, P_D, of meeting a LCD, $\sigma^2(T_E)$, t_e, and $\sigma^2(t_e)$. The report may list the predicted critical path, and several other paths in increasing order of K_E. Management may take appropriate action by reallocation of resources if the probability of meeting the overall project LCD is unacceptably small. In order to "buy time" additional resources are allocated to, and additional managerial attention devoted to those activities on the predicted critical path, and to those whose mean slack is relatively small. Resources may be shifted from paths with greater K_E to paths with lesser K_E. In some cases reorganization of the network is required, placing some activities in parallel rather than in series paths, or eliminating activities whose products are not entirely essential to the project.

The results of original PERT computations may be the basis for establishing an overall project schedule. This schedule specifies scheduled completion times, t_s, for activities, and scheduled times of occurrence, T_S, for certain events.

While the project is in progress, revised inputs are submitted periodically, showing percent of completion, and revised estimates. In the case of a partially completed activity, these estimates should be considerably more reliable than those submitted before work was commenced. These inputs are fed to the computer, and another network calculation is run, as before, except that completed activities are noted and the actual completion time is used in the network rather than an estimated time. The results of these runs may necessitate schedule changes, and/or any of the actions previously mentioned.

4. Critique of the Basic PERT System

The system which we have just described tends to identify and call attention to a single path, the Predicted Critical Path. This path, though actually uncertain, is deterministically established by virtue of several assumptions. In most networks, the activities on this "critical path" will have probabilities of actually being critical much less than unity. Furthermore, other activities, not on this path, may have probabilities of being critical larger than some activities on the "critical path." [18] Hence, the system, by focussing attention on the single path, may deemphasize activities which are actually more important.

The assumptions which were made in determining T_E and T_L lead to optimistic results in these figures, and in k_E. By reason of the same assumptions, the probability that an event is completed before a BCD is always optimistic.

T* is the random variable representing the elapsed time to occurrence of an event. Suppose there are n different paths to the event. Let the random variables, $p_1^*, p_2^*, \cdots, p_n^*$, be the lengths of these n paths. Now T* is the maximum of these n variables, but the maximum of a set of random variables is not normally distributed, even if the individual variables themselves are normal. In particular, T* does not have the same distribution as the p* with the largest mean, nor does T* have the same mean as this p*. Furthermore, the p_i^* are neither independent nor identically distributed, hence the calculation of the distribution of the maximum of this set is so complicated as to render it infeasible by analytic methods.

In general, the probability that an event occurs before some

specific scheduled date is the probability that none of the possible paths overrun the scheduled date. By the PERT assumptions, we considered only one path, namely that with the greatest mean length. Obviously, if there are several parallel paths with nearly the same mean length, then by considering only the probability that one of the paths does not overrun, we are being quite optimistic in our result.

For example, suppose there are three parallel and independent paths, each approximately equally likely to be the critical path, and each having a probability of being less than D_s of about 0.50. Then P_D, the probability that the project is completed prior to the DCD, is the probability that none of the path lengths exceed D_s. Hence, $P_D = (.5)^3 = .125$. The PERT procedure would consider only the path with the largest mean length, and would calculate $P_D = 0.50$. For networks with several parallel paths of comparable length, the error would be quite large, as in the example. In networks with only one predominantly long path, the error would be negligible.

In order to obtain a more general prediction of criticality than the single path prediction, we need to calculate the probability that each activity will be on the critical path. We shall call this probability the <u>Criticality Index</u>, written, P_C. Computation of P_C involves the concept of determining the probability that a particular p_k* will be the minimum of a set of n random variables, $\{p_i^*\}$. As before, these p_i* are not independent nor identically distributed, hence this computation is not feasible analytically. It is not, however, difficult to determine P_C by the <u>Monte Carlo</u> method. This method also readily yields T_E, T_L, and P_D without resorting to the assumptions which were so

troublesome, but necessary, in the analytic approach. A detailed discussion of this method, and suggestions for specific use of the criticality index will form a substantial portion of this study.

The PERT Beta probability distribution was chosen as the distribution for t* for an activity. Recall that in specifying this distribution the variance of the distribution was arbitrarily chosen to be $\left(\frac{P - O}{6}\right)^2$. By this assumption, we have specified the degree of uncertainty with which the estimator makes his three estimates. In other words, we are denying the difference between the case where the estimator may predict, with a high degree of confidence, that t will occur within a few time units of M, and the case where t has a higher likelihood of occurring near the extremes of the range of t*. The probability distribution for the former case should have a shape similar to that shown in Figure 5, while the latter case would be better represented by Figure 6. It is not unreasonable to assume that, under certain conditions supervisors will have a basis for estimates with a narrow uncertainty band about the mode, even though the range of possible values of t* might be quite large. Under other conditions, the supervisor might be unable to place M with an appreciable degree of confidence. In the interest of more accuracy in predicting the system behavior, perhaps a choice of density functions should be available to the estimator to help describe his confidence in his estimates. This would not require a knowledge of probability theory on the part of the supervisor. He could simply be asked to rate his uncertainty as to the relative position of M on a numerical scale, with three descriptive choices. His choice would lead to a corresponding probability distribution. Appendices I, II, III and IV

describe means of providing this flexibility in choice of density function with the desired degree of variance.

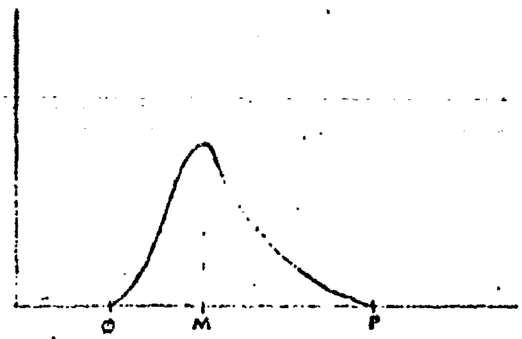


Figure 5

A Density Function with Small Variance

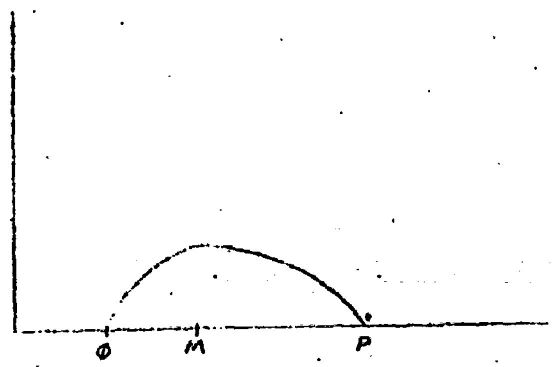


Figure 6

A Density Function with Large Variance

5. The Monte Carlo Technique

We have seen that the completion time, t*, for any activity in a prepared project is a random variable. As the project is performed, each activity is completed in some actual time, t. Only after completion of the entire project can actual values of t, slack, and elapsed time, T, for events and activities be fixed, and the critical path determined.

The Monte Carlo method calculates average values of these parameters by repeatedly simulating the performance of the entire network, and statistically averaging the results. The simulation of the project is accomplished as follows:

A random value of t for each activity in the network is drawn from the probability distribution for t* for the activity. These random values of t, called realizations of t*, are drawn in such a way that in repeated drawings from a specific probability distribution, the number of realizations falling within a segment of fixed length on the t axis is proportional to the height of the probability density curve over that segment. Each activity has its own unique probability distribution, determined by the three estimates, hence, for one simulation of the project [illegible] realization of t* drawn for an activity is a completion time which [illegible] resulted from a normal performance of the activity. One simulation of the project consists of drawing a value of t for each activity of the network. Calculations of the network are then performed, using [illegible] values of t instead of the t_e used in the analytical method. T and [illegible] activity are calculated, and then the critical path for that [illegible]ion is determined. For each activity a tally is incremented

if the activity was actually on the critical path. Another tally is incremented for each scheduled event if the event was accomplished prior to its scheduled completion date.

The above simulation is repeated many times and the values of t, T, and K for each activity are added to cumulative totals after each simulation of the project. After an adequate number of replications, these cumulative totals and tallies are divided by the number of replications, yielding the average values, t_e, T_E, and K_E for each activity and the probability that the activity will be on the critical path. For all scheduled events, the probability that the event was completed prior to the scheduled date is also computed. The observed variance of these statistics is also available if desired, as well as an approximate probability distribution for project duration.

The method of sampling the probability distributions of the activity times for realizations of $t*$ is discussed in detail in Appendices I, II, III and IV. Using the above techniques, it is not difficult to provide a choice of probability distributions for $t*$ in order to account for varying degrees of uncertainty in estimates, as was discussed earlier.

The computer codes designed to implement the basic PERT system may be modified, without great difficulty, to perform the Monte Carlo calculation. The author recently modified the North American Aviation PERT Mark III code in this manner. This program, designed for the IBM 7094, will accommodate a network of 1500 activities without requiring external memory units. [16]

The Monte Carlo technique is not subject to the pitfalls of independence, assumptions of normalcy due to the Central Limit Theorem, and

assumption of a particular longest path, etc., which plague the analytic computation. The inaccuracies inherent in the Monte Carlo technique are due to non-randomness in the random number generator, and perhaps, failure to perform a sufficient number of replications to permit the law of large numbers to obtain. Excellent random number generators are available, providing results well within the limit of accuracy imposed by the time estimates. Statistical methods exist by which to determine the number of replications required in order to obtain the desired degree of confidence in the results. [17]

The system is limited in accuracy only by the human error in estimating activity times, and in approximating the correct probability distribution for us.

The Monte Carlo technique requires more computer running time than the analytic method. An efficiently coded routine requires approximately 750 microseconds to draw a random number and, from the calculated [illegible], to transform it into a realization of [illegible]. For details of [illegible] employed, see Appendix I. The approximate increase in [illegible] time required, if the Monte Carlo technique is used in lieu of the standard analytic method, may be estimated by the product:

(No. of activities)(No. of replications)(750×10^{-6} sec).

For example, a network consisting of 1000 activities would require [illegible] hours of additional computer running time if a Monte Carlo calculation of 1000 replications were used instead of the standard technique. [illegible] inherent in this technique would seem to render this increase in running time negligible.

6. The Extension to PERT/COST [10]

An extension of the basic PERT system, PERT/COST, was adopted 1 July [illegible] as the basic standard for management time and cost control systems by DOD and NASA. PERT/COST extends the original PERT system to provide cost estimation and cost monitoring features in addition to the procedure for prediction and control of time performance. Two additional optional features, a Time-Cost supplement, and a Resource Allocation supplement were also promulgated.

PERT/COST adds to the basic system the following capabilities:

a. Initial estimation of project cost, broken down by work packages, sub-systems, etc.

b. Provision of a consolidated estimate of the requirements for manpower and other resources as an aid to scheduling and procurement.

c. Provision of in progress cost reports, showing current cost status and revised cost estimates in comparison with budgeted expenditures and contract estimates.

The optional Time-Cost supplement provides a method of estimating three feasible schedules for project completion, with their estimated costs and an evaluation of the technical risks associated with each schedule. These schedules are designed to provide completion in accordance with three criteria: (1) the most efficient use of time and resources; (2) completion by the DCD; (3) earliest feasible completion.

The Resource Allocation supplement outlines a procedure for scheduling a project to meet a DCD in an optimal manner with respect to costs, and utilization of manpower.

The concept of operation of PERT/COST is essentially as follows:

A. Project breakdown. The project is broken down into activities and work packages, and graphically represented by a network. For the purposes of cost reporting, estimating, and accounting, several activities may be combined into a "work package", when the activities involved are small, and detailed cost reporting and estimating of these activities would involve unnecessary expense. In general, a work package does not comprise a closed network. That is, it may not be integrated and represented by a single activity due to the presence of events interior to the work package which are connected by activities to other work packages. In this case, the work package must be broken down into its component activities and events for the network time calculations. The cost estimating and expenditure reporting section of the system may utilize the entire work package as the smallest organizational division.

Certain work packages may not appear on the project network. In general, these are the functions of management not directed toward the accomplishment of specific activities and events. For example, the accounting, purchasing, or management travel costs for the project would be difficult to represent as activities in the network, however, the cost of these work packages may be estimated and reported, and treated in the cost analysis portion of the PERT/COST system.

B. Time calculations. Time estimates are made for each activity, and the network calculations performed. The project duration is compared with the DCD, and if necessary, any of the following actions taken in order to reduce the overall project duration:

(a) Reorganize the network by increasing the parallelism of activities.

(b) Commit additional resources to selected activities, or reallocate resources within the network.

(c) Change or delete activities.

C. Manpower loading. [9] A breakdown by skills of manpower required for an activity/work package may be included with the time and cost estimates. Computer programs now in existence assemble this information and prepare a report and display showing the project manpower requirements, by skills, plotted against time. [7, 11] This information provides project management with a forecast of manpower needs, and a basis for rescheduling activities in order to best utilize available manpower. Scheduling a large project without considering the overall use of manpower skills leads to very uneconomical manpower use. One time period may require a manpower level for a certain skill far in excess of that available, while the following time period makes little use of that skill. In performing the necessary rescheduling, the time constraints of the network and criticalness of activities must be considered. Those activities least likely to be critical are the obvious candidates for rescheduling for the purpose of leveling manpower requirements. Computer programs now in existence, supplementing standard PERT routines, perform this rescheduling under the constraints of maximum availability of manpower and with consideration of slack. [7, 11]

D. Scheduling the network. A scheduled duration, t_s, is promulgated for each activity. These times may be less than, equal to, or greater than t_e, depending on the situation and management policy. Using the t_s for each activity, the network is computed and values of S_E and S_L, corresponding to T_E and T_L, determined. S_E represents the earliest

date on which an activity/event can be scheduled for completion. S_L represents the latest date on which an activity/event may be scheduled for completion without causing a schedule slippage for the project completion. S_E and S_L are computed in exactly the same manner as were T_E and T_L, when t_s is used as activity time instead of t_e. As a final step in scheduling, an elapsed time, T_S, is imposed at each event. T_S is chosen to lie between S_E and S_L, and may be translated into a firm scheduled completion date for activities immediately preceding the event, and a beginning date for activities immediately following the event.

3. Budgeting the project. After the schedule is finalized, cost estimates are prepared for each work package, and a budget prepared for allocating funds.

4. Monitoring progress of the project. As the project progresses, reports are submitted from activities/work packages, indicating progress, new time estimates, funds committed to date, and new estimates for cost. This information forms inputs to the PERT/COST computer program, which performs the network calculations, and prepares reports showing new values for T_E, T_L, S_E, P_D, and the new critical path, as well as current expenditures in comparison with estimates and budget. The new projected cost curve is also generated and compared with estimates and budget. With this information, management may reschedule, reallocate, re-budget, or take whatever action as may be appropriate to the situation.

The Time-Cost Option Supplement. The purpose of this supplement is to standardize procedures by which a project manager may prepare three alternative project schedule proposals, and evaluate the technical risks associated with each. By technical risk, we mean the gamble of

performance, time, and cost incurred by departing from the best development technique in order to meet the schedule.

First a plan is formulated designed to meet the project ECD, as imposed by the controlling agency.

Next, a _Most Efficient_ plan is formulated, in which the best development techniques are used in order to reduce technical risks, and most efficient use is made of manpower and resources, resulting in greater duration for the project. Total cost is generally lower for this plan.

Finally, a _Shortest Time_ plan is proposed. The purpose of such a plan is to explore the feasibility of scheduling the project for a duration shorter than that proposed by the contracting agency in order to benefit from any strategic benefits which might accrue from such early completion. In order to arrive at the shortest time plan, the schedule is compressed by allocation of additional resources, paralleling activities which should normally go in sequence, eliminating activities, and changing technical approaches to the problems of design and construction. All such actions serve to increase the technical risks involved, and most of them increase cost.

The three options are presented to the contracting agency with estimates of cost and time, and with a comparison and evaluation of the technical risks associated with each plan.

The PERT/COST system is the primary tool for obtaining the time and cost estimates needed to prepare the three schedule proposals.

The Resource Allocation Supplement. This supplement provides a procedure by which project managers may schedule the project in the optimal manner with regard to the time-cost trade-off. The concept of operation

is as follows.

For any activity there may exist several feasible schedules, involving different time-cost relationships. (See Figure 7.) Efficient use of manpower and existing resources and machinery may result in schedule A. Further time extension to B might result in higher costs due to the effect of fixed costs. By use of overtime, hiring additional manpower, or purchase of additional machinery or space, time-cost combination C or D might be achieved.

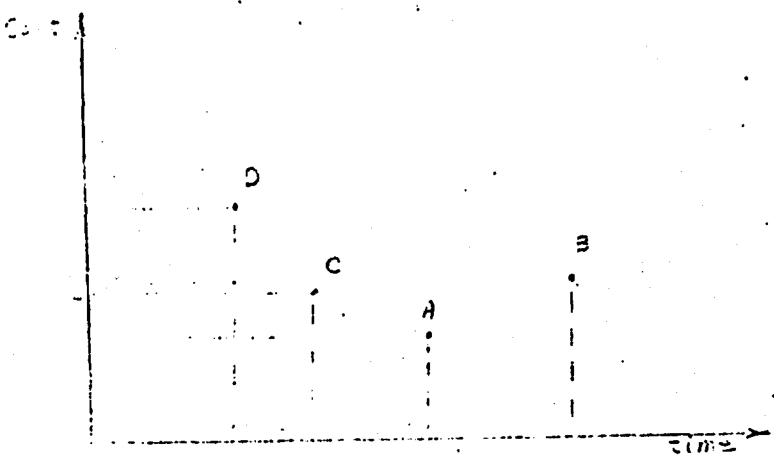


Figure 7

Time-Cost Plans for an Activity

Schedule A would be the most economical for this activity, hence would be the time estimated for the activity for the first network calculation. In a similar manner, most economical estimates are obtained, with corresponding cost estimates, from the other activities. The network calculation is then performed, and the project duration, D, compared

with D_S. If $D > D_S$, selected activities on the critical path must be shortened by resorting to a higher cost-shorter time work schedule.

For each activity on the critical path, determine the increase in cost, divided by the decrease in time that would occur in moving to each shorter time-cost point. This is the slope of the time cost curve for the corresponding time reduction. The activity with the lowest value of slope is then chosen as the activity to reschedule.

If this time reduction in the critical path exceeds the difference in slack between the critical path and the smallest slack not on the critical path, it becomes necessary to recompute the network and determine a new critical path. The above process is repeated as often as necessary until $D \leq D_S$.

Slack paths are then reexamined to see if any activities may be extended to lower cost points without going critical. If fixed costs were not a part of the activity estimates, notice may be taken of the fact that the most economical point to operate is usually to the left of the minimum cost point with respect only to direct costs. After determining the most economical plan with respect to direct costs, a test is made to determine if further time reduction below the DCD would result in lower overall costs. This is done by adding on the fixed costs for the project and attempting further reduction in time. If further time reduction results in lower overall cost, then further time reduction is performed until total costs reach a minimum.

The schedule is then adjusted as necessary to reach an optimum leveling of manpower and other resources within the existing constraints and considering total cost.

7. A Method for Resource Allocation, using the Criticality Index

In Section 4, we pointed out the errors in the concept of a deterministic prediction of the critical path. We also showed the optimistic bias inherent in the assumption that the longest path to an event will be the path whose mean length is greatest. The Monte Carlo technique does not depend on this assumption, hence, with this technique we can calculate P_C, T_E, and P_D with accuracy limited only by the supervisor's estimates and our approximation of the true probability distributions of t for the activities. We shall now develop a system for optimum resource allocation utilizing the criticality index obtained from the Monte Carlo calculation.

In order to use this method, we must require that a work package be a closed network. A closed network is a network with a single beginning event and a single end event. Activities exterior to the closed network must not be incident to, or emanate from, events within the closed network other than the beginning or end event. (See Figure 8.) In this example, if activities not included in the work package should be incident to, or emanate from events B, C, or D, then the network would not be closed. This same restriction is implicit in the resource allocation supplement of the standard PERT/COST system, although not stated in the PERT/COST document. Notice that the closed network may be integrated and represented by a single activity connecting events A and E. Some PERT computer programs now in existence are capable of integrating such subnetworks for simplification of the network for the benefit of higher levels of management. [15]

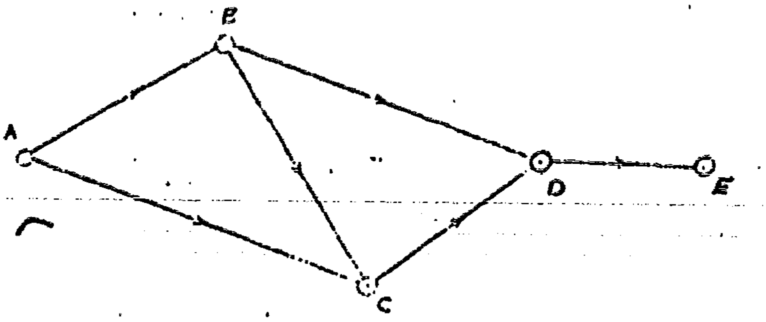


Figure 8

A Closed Network

We next require that the supervisor in charge of each work package submit as many time-cost plans as are feasible for his work package completion. These plans are represented by points on the time-cost curve in the same manner as previously discussed in connection with the Resource Allocation supplement. No points to the right of the minimum cost point are submitted. The time-cost information is represented as a set of n pairs, (time,cost), in descending order of time and corresponding ascending order of cost. For example:

14 weeks, $40,000

12 weeks, 60,000

10 weeks, 90,000

The time corresponding to the minimum cost point is taken as the most likely estimate, M, of activity completion time. The additional time estimates, O and P, are also submitted for the minimum cost plan.

These three estimates and the corresponding cost are the basis of the preliminary calculation of the network. From this calculation, the probability, P_D, of meeting the project ECD, is determined. Recall that this probability is considerably more valid as a result of the Monte Carlo technique.

If P_D is unacceptably small, the network must be compressed by re-designing the network layout and/or allocating more resources to selected work packages in order to achieve shorter completion time. If, after whatever network redesign is considered feasible, the network must be further compressed, the resource allocation phase of the program is performed. This phase consists of the following logical steps:

1. For each time-cost plan of each activity having positive P_C, the following figure of merit is computed.

$$ETB = P_C \frac{\Delta t}{\Delta C} \tag{4}$$

where Δt is the reduction in time achieved by using the T-C plan being considered, over that used in the previous network calculation. ΔC is the corresponding increase in cost. ETB may be considered as the <u>expected</u> amount of time which will be "bought" per unit cost by selecting the T-C plan being considered.

2. Form 3-tuples consisting of each ETB, its corresponding Δt, and the activity identification. Order this list in descending order of ETB.

3. Select from the top of the list, a sufficient number of reduced T-C plans so that the sum of these Δt's does not exceed a predetermined percentage, p, of D. The best value for p may be determined experimentally. The selection is made from the top of the list, but only one

plan, that with the largest Δr, is retained for any one activity.

4. For the activities chosen for time reduction, the time corresponding to the new T-C plan is taken as the new value of M. The estimates for P and O are revised downward by multiplying by the ratio, $\frac{M\ (new)}{M\ (old)}$. (This procedure may be subject to question; another method would be to require P and O estimates for each T-C plan.)

5. With the new time and cost estimates for the accelerated activities, the network is again calculated, and P_D determined. All activities will have a new P_C. The mean duration, and P_D are again evaluated. If still unacceptable, the process may be continued until either an acceptable P_D is attained, or the budgetary limit is reached. The schedule is finalized on the basis of the time plans used in the last network calculation, after consideration of manpower and other resource leveling.

By the above method, resources are allocated where, probabilistically, they can be expected to contribute most effectively to shortening project duration. The requirement for several feasible time-cost plans may serve the additional purpose of forcing line supervisory personnel to consider the time-cost relationship more carefully, allowing operation at the most efficient point on the time-cost curve whenever feasible.

An optimum value for p can be determined with a few experimental runs of the program. p should be chosen as large as possible in order to decrease the number of network calculations required, but small enough so that successive calculations produce a smooth compression of the network. Operation of the program will tend to decrease the differences between the MTB's for the activities, which is equivalent to operating the activities at the same level of expected marginal utility. This is

a well known principle of optimization.

The inputs to the system, three time estimates and a T-C function specified by several 2-tuples, are not complicated, and are compatible with the concept of progress and cost monitoring envisioned in the TIME/COST system.

A flow diagram of the logic used in this resource allocation method is shown in Appendix V.

8. A Military Application of the PERT Network Approach

One of the most outstanding features of the PERT approach is the graphical portrayal of the project by means of the network. By this device, relationships between the various tasks may be clearly visualized. The insight thus gained by the managers of the project, into these complex relationships, enables them to function with greatly increased effectiveness in their managerial capacity.

A military operation is in many ways similar to a development project. The operation may be divided into a number of tasks, with many interrelationships existing between individual tasks. Hence, the network may serve the same useful function in this application as in industrial development.

The phases of military planning known as "The Development of the Plan," and "Supervision of the Planned Action," are most readily benefited by the network approach. In the formalized military planning sequence, the Development of the Plan phase has been preceded by the "Estimate of the Situation," in which the assigned mission has been studied, possible outcomes based on enemy capabilities and alternative own courses of action have been analyzed in a game theoretic matrix, and a decision has been reached regarding a general course of action to be pursued.

The problem now involves developing a complete, detailed plan for the operation, including organizing the available forces and assigning to each task unit the appropriate tasks which make up the general plan. At this stage, detailed planning is done in which training, acquisition of intelligence, movement, communications, logistics, and battle action are

all considered.

The plan begins with an initial concept, which may be represented by a relatively simple network, whose tasks are stated in broad terms of accomplishment. As the planning becomes more detailed, these initial tasks may be subdivided into networks of lower-order tasks. This phase may be done at lower echelons of command, upon receipt by them of their superior's directive.

The network will aid in establishing a firm schedule for the operation, in much the same manner as is done in industrial projects. It may also be a valuable aid for conducting briefings for subordinates, since it will enable them to better appreciate the relationship of their own tasks to the whole operation.

During the operation, the network may serve as a display, on which current progress is indicated as reported. The effects of delays and failures may be more readily evaluated and corrected.

The initial concept of a plan to execute an amphibious landing in enemy-held territory might be represented by the network shown in Figure 9.

A network may also be used to lay out the basic strategy of a projected military campaign. In this usage, tasks would be represented by subsidiary operations, such as: "Seize Island B," or "Cut supply line [illegible]." The procedure is similar to that employed in the operation just discussed, except that it takes place over an extended period of time. In many cases, proposed actions are tentative, or the selection of alternative actions may depend on future developments, such as the success or failure of a previous endeavor, or on the subsequent enemy reaction. We may indicate this need for decision at a particular point in the network

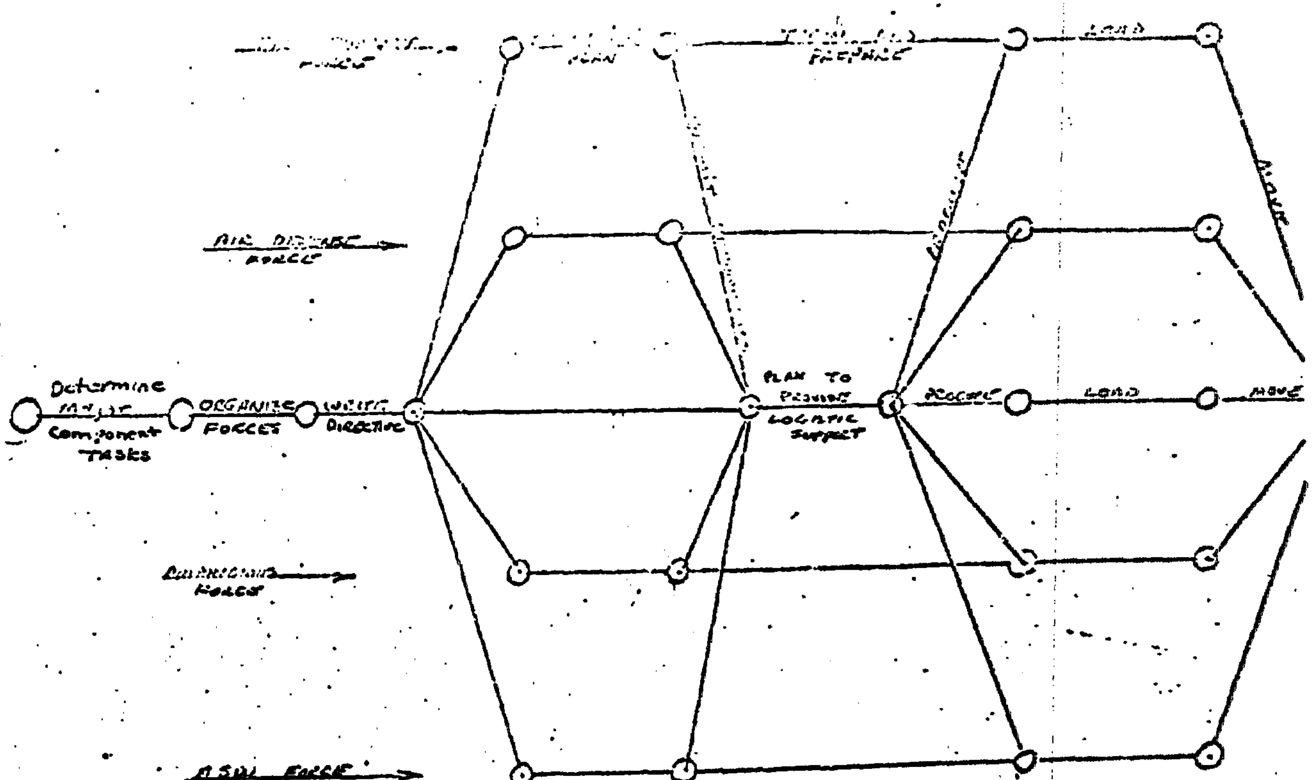
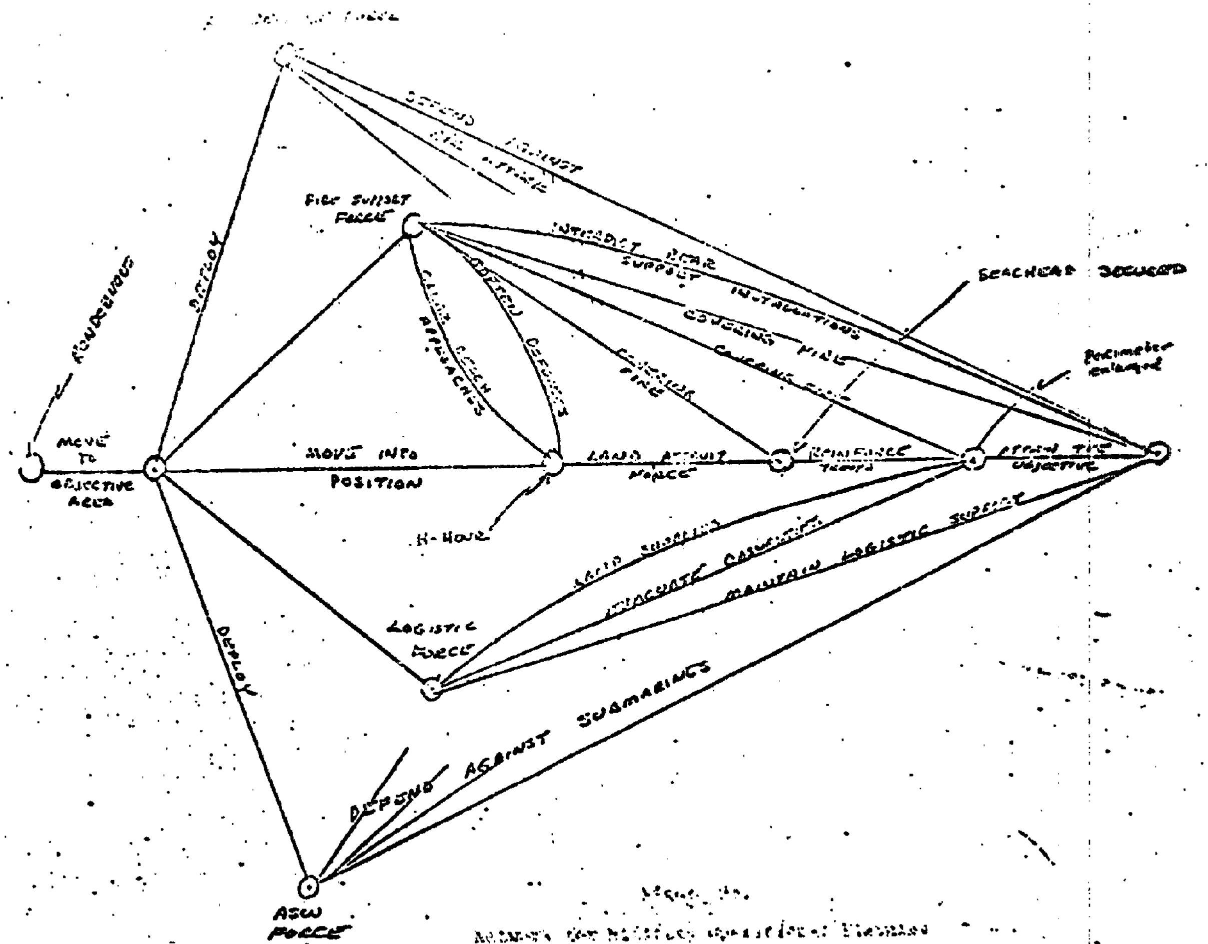


Figure 9A.

Network for Military Operational Planning

(Continued to Fig. 9B)



Network for Military Operational Planning

38

by showing a "decision box" in lieu of an event at that point. The decision box has more than one task emanating from it; the decision, to be made at some later date, determining which task(s) to be performed. The network then proceeds to show the planned action to be taken for each alternative. Figure 10 is an example of a portion of such a strategy network. [13,14]

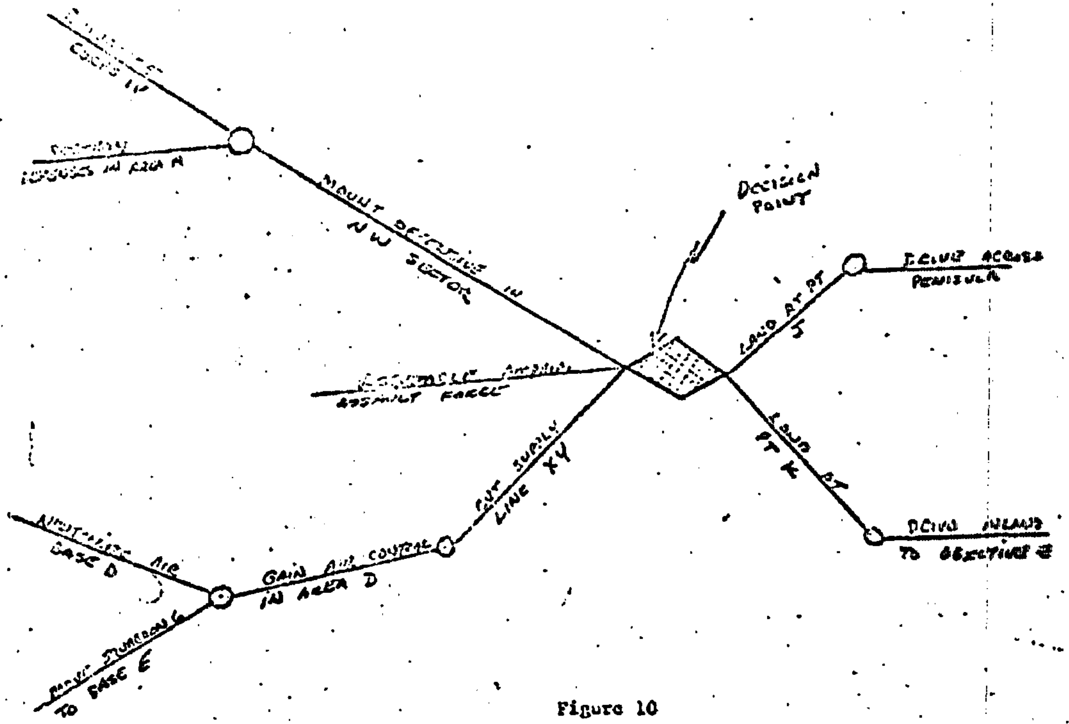


Figure 10

Network for Strategic Planning

APPENDIX I

Monte Carlo Techniques with the Beta Distribution

The model for activity completion time, t^*, chosen by the originators of PERT, is the Beta probability distribution, characterized by its density function: [2]

$$f_{t^*}(t) = K(t-\emptyset)^{\alpha}(P-t)^{\gamma}, \quad \emptyset \leq t \leq P \tag{5}$$

$$= 0, \qquad \text{elsewhere.}$$

We may reduce this to the standard form of the Beta distribution by the following transformation:

$$x^* = \frac{t^* - \emptyset}{P - \emptyset} \tag{6}$$

The probability density function of x^* is:

$$f_{x^*}(x) = K_1\, x^{\alpha}(1-x)^{\gamma}, \quad 0 \leq t \leq 1 \tag{7}$$

The variable, x^*, is equivalent to t^*, where $\emptyset = 0$ and $P = 1$.

The two parameters, $\emptyset$ and P, specify the upper and lower limits of the variation of t^*. It is now desired to specify the remaining two parameters, α and γ, so that the mode of the density function occurs at M, and so that the distribution has the desired variance.

Let $m = \frac{M-\emptyset}{P-\emptyset}$, denote the mode of x^*. By finding the root of $f'_{x^*}(x)$, we obtain:

$$m = \frac{\alpha}{\alpha + \gamma} \tag{8}$$

The variance of x^* is given by:

$$\sigma^2(x^*) = \frac{(\alpha + 1)\,(\gamma + 1)}{(\alpha + \gamma + 2)^2\,(\alpha + \gamma + 3)} \tag{9}$$

Now suppose we desire that the standard deviation, $\sigma(t^*)$, be expressed by $\left(\frac{P-\emptyset}{d}\right)$, where d is an arbitrary constant. Then $\sigma(x^*) = \frac{1}{d}$. By

requiring that $\sigma^2(x^*) = \frac{1}{d^2}$, we obtain the following relationship in α and m.

$$\alpha^3 + (d^2m^3 - d^2m^2 + 7m)\alpha^2 + (16 - d^2)m^2\alpha^2 - (d^2 - 12)m^3 = 0 \tag{10}$$

Now α can be determined as a function of m and d by the above equation. With α determined, γ is specified by solving equation (8):

$$\gamma = \frac{\alpha(1-m)}{m} \tag{11}$$

The mean of x* is given by:

$$E[x] = \frac{\alpha + 1}{\alpha + \gamma + 2} \tag{12}$$

Since α must be determined from the cubic equation, (10), for each value of m, the computation of E[x] is cumbersome. In the PERT Beta model, d was chosen as six. For this value of d, E[x] is approximately linear in m, and may be approximated by the following relation:

$$E[x] = \frac{4m + 1}{6} \tag{13}$$

The transformation to t* yields:

$$E[t] = \frac{0 + 4M + P}{6} \tag{14}$$

<u>Random Sampling from the Beta Distribution</u>. There are several good random number generating routines which yield random numbers from the uniform (0,1) distribution. We shall use the technique of the Probability Integral Transformation to transform the random number, u, drawn from the uniform (0,1) distribution, to a corresponding sample from whatever other probability distribution we desire. [3]

This transformation depends on the following theorem:

For any random variable, v^*, having the probability distribution function, F_{v^*}, and the density function, f_{v^*}, define the random variable, $u^* = F_{v^*}$, that is:

$$u = F_{v^*}(v) = \int_{-\infty}^{v} f_{v^*}(t)\,dt \tag{15}$$

then, u^* is a uniformly distributed random variable on the interval (0,1). Hence, we may use the inverse function, $v = F^{-1}(u)$, to transform the sample, u, from the uniform (0,1) distribution to the corresponding sample, v, from the desired distribution.

The distribution function:

$$F_{x^*}(x) = K_1 \int_{0}^{x} y^{\alpha}(1-y)^{\gamma}\,dy \tag{16}$$

for the Beta variable, x^*, must be calculated by numerical integration. In order to utilize the procedure given above for sampling from this distribution, we may store the tabled distribution functions in the memory of the computer and use a table look-up routine to enter the table with the random sample, u, and determine the corresponding value for x. A complete distribution function must be stored for each of the increments of m, and for a limited number of values of d.

The input card for an activity specifies the choice of d which best represents the estimator's uncertainty. Also specified are the three estimates, O, M, and P. With d and M specified, the proper table may be selected. u is produced by the random number generator. The corresponding value of x is obtained from the table, then transformed into a realization of t^* by the following transformation:

$$t = (P - \emptyset)\,x + \emptyset \qquad (17)$$

Tables I through V are tables of distribution functions for the standard Beta distribution, with d taking on values of 4 through 8, respectively. A complete function is given for each of 11 values of m.

As an example of the use of the procedure given above, suppose $\emptyset$, M, and P were 12, 15, and 20 weeks, respectively, and the selection, $d = 6$, is made. Then $m = \frac{15-12}{8} = .375$. Suppose we draw the random number, .795, from the uniform random number generator. Entering Table III for $m = .3$, with the argument, .795, we obtain $x = .50$. From the table for $m = .4$, linear interpolation yields $x = .572$. Interpolating between these two values of x, we obtain $x = .554$ for $m = .375$. The corresponding realization of t^* is then $8x + 12 = 16.4$.

An efficiently coded routine designed to perform the above table searches and calculations, requires approximately 750 microseconds.

Figures 11, 12, and 13 are families of curves of the probability density functions for x^* corresponding to $d = 4, 6,$ and 8, respectively.

DISTRIBUTION FUNCTION OF THE STANDARD BETA FUNCTION WITH D = 4

X	A= .000 G= .720 M= .000	A= .079 G= .714 M= .100	A= .174 G= .694 M= .200	A= .280 G= .554 M= .300	A= .393 G= .589 M= .400	A= .500 G= .500 M= .500
.05	.071	.059	.047	.035	.025	.018
.10	.068	.062	.054	.045	.037	.029
.15	.233	.207	.179	.149	.119	.093
.20	.309	.280	.247	.211	.175	.142
.25	.381	.352	.316	.276	.234	.195
.30	.451	.421	.384	.342	.296	.252
.35	.516	.487	.451	.407	.360	.312
.40	.579	.551	.515	.473	.424	.373
.45	.637	.611	.578	.535	.488	.436
.50	.692	.669	.638	[illegible]	.552	.500
.55	.743	.722	.694	.658	.614	.564
.60	.790	.772	.748	.715	.674	.627
.65	.833	.818	.797	.769	.732	.688
.70	.872	.860	.843	.819	.787	.748
.75	.907	.897	.884	.865	.839	.805
.80	.936	.930	.920	.905	.885	.858
.85	.961	.957	.951	.941	.927	.907
.90	.981	.979	.975	.970	.961	.949
.95	.994	.994	.992	.990	.987	.982
1.00	1.000	1.000	1.000	1.000	1.000	1.000

X	A= .590 G= .393 M= .600	A= .654 G= .280 M= .700	A= .694 G= .174 M= .800	A= .714 G= .079 M= .900	A= .720 G= .000 M=1.000
.05	.013	.010	.008	.006	.006
.10	.023	.018	.015	.013	.012
.15	.073	.059	.049	.043	.038
.20	.115	.094	.080	.070	.063
.25	.161	.135	.116	.102	.092
.30	.213	.181	.157	.140	.126
.35	.268	.231	.203	.181	.164
.40	.326	.285	.252	.227	.207
.45	.386	.342	.306	.277	.253
.50	.448	.402	.362	.331	.304
.55	.512	.464	.422	.388	.358
.60	.576	.527	.484	.448	.415
.65	.640	.593	.549	.512	.477
.70	.704	.658	.616	.578	.541
.75	.766	.724	.684	.647	.610
.80	.825	.789	.753	.718	.681
.85	.881	.852	.821	.791	.756
.90	.932	.911	.889	.865	.834
.95	.975	.965	.953	.938	.916
1.00	1.000	1.000	1.000	1.000	1.000

Table I

$$F_{X}(x) = \frac{\int_0^x y^A (1-y)^G \, dy}{\int_0^1 y^A (1-y)^G \, dy}$$

M is the position of the mode

DISTRIBUTION FUNCTION OF THE STANDARD BETA FUNCTION WITH D = 5

X	A= .000 G=1.826 M= .000	A= .217 G=1.951 M= .100	A= .509 G=2.036 M= .200	A= .870 G=2.030 M= .300	A=1.262 G=1.093 M= .400	A=1.625 G=1.625 M= .500
.05	.114	.078	.044	.019	.007	.002
.10	.104	.088	.066	.041	.022	.010
.15	.353	.208	.210	.132	.072	.036
.20	.455	.389	.304	.210	.129	.072
.25	.546	.484	.398	.296	.197	.120
.30	.626	.570	.489	.385	.275	.181
.35	.697	.648	.575	.475	.360	.251
.40	.758	.718	.653	.561	.447	.329
.45	.811	.778	.724	.643	.534	.413
.50	.856	.830	.787	.717	.619	.500
.55	.893	.874	.840	.784	.698	.587
.60	.923	.910	.885	.841	.770	.671
.65	.947	.939	.921	.889	.833	.749
.70	.966	.961	.949	.927	.887	.819
.75	.980	.977	.970	.956	.929	.880
.80	.989	.988	.985	.977	.961	.928
.85	.995	.995	.993	.990	.982	.964
.90	.998	.998	.998	.997	.994	.987
.95	1.000	1.000	1.000	1.000	.999	.998
1.00	1.000	1.000	1.000	1.000	1.000	1.000

X	A=1.894 G=1.263 M= .600	A=2.030 G= .870 M= .700	A=2.036 G= .509 M= .800	A=1.951 G= .217 M= .900	A=1.826 G= .000 M=1.000
.05	.001	.000	.000	.000	.000
.10	.005	.002	.002	.001	.001
.15	.018	.010	.007	.005	.005
.20	.039	.023	.015	.012	.011
.25	.071	.044	.030	.023	.020
.30	.113	.073	.051	.039	.033
.35	.166	.111	.079	.061	.051
.40	.230	.159	.115	.090	.075
.45	.302	.216	.160	.126	.105
.50	.391	.283	.213	.170	.141
.55	.466	.357	.276	.222	.185
.60	.553	.439	.347	.282	.236
.65	.640	.525	.425	.352	.296
.70	.725	.615	.511	.430	.365
.75	.803	.704	.602	.516	.444
.80	.871	.790	.696	.611	.532
.85	.928	.868	.790	.712	.632
.90	.969	.934	.879	.817	.742
.95	.993	.981	.956	.922	.865
1.00	1.000	1.000	1.000	1.000	1.000

Table II

$$F_x(x) = \frac{\int_0^x y^A \, (1-y)^G \, dy}{\int_0^1 y^A \, (1-y)^G \, dy}$$

M is the position of the mode

DISTRIBUTION FUNCTION OF THE STANDARD BETA FUNCTION WITH D = 6

X	A= .000 G=2.872 M= .000	A= .361 G=3.252 M= .100	A= .891 G=3.565 M= .200	A=1.574 G=3.673 M= .300	A=2.320 G=3.480 M= .400	A=3.000 G=3.000 M= .500
.05	.153	.090	.035	.008	.001	.000
.10	.131	.109	.071	.032	.010	.002
.15	.449	.350	.222	.107	.040	.012
.20	.564	.472	.338	.195	.088	.033
.25	.661	.581	.454	.298	.159	.071
.30	.740	.675	.563	.408	.248	.126
.35	.805	.755	.662	.519	.350	.200
.40	.857	.820	.747	.624	.458	.290
.45	.898	.872	.818	.718	.567	.392
.50	.929	.912	.874	.798	.670	.500
.55	.953	.943	.917	.863	.761	.608
.60	.970	.964	.949	.913	.838	.710
.65	.982	.979	.971	.949	.898	.800
.70	.990	.989	.985	.973	.942	.874
.75	.995	.995	.993	.987	.971	.929
.80	.998	.998	.997	.995	.988	.967
.85	.999	.999	.999	.999	.996	.988
.90	1.000	1.000	1.000	1.000	.999	.997
.95	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

X	A=3.481 G=2.321 M= .600	A=3.672 G=1.574 M= .700	A=3.565 G= .891 M= .800	A=3.252 G= .361 M= .900	A=2.872 G= .000 M=1.000
.05	.000	.000	.000	.000	.000
.10	.001	.000	.000	.000	.000
.15	.004	.001	.001	.001	.001
.20	.012	.005	.003	.002	.002
.25	.029	.013	.007	.005	.005
.30	.058	.027	.015	.011	.009
.35	.102	.051	.029	.021	.017
.40	.162	.087	.051	.036	.029
.45	.239	.137	.083	.057	.045
.50	.330	.202	.126	.088	.068
.55	.433	.282	.182	.128	.099
.60	.542	.376	.253	.180	.138
.65	.650	.481	.338	.245	.189
.70	.752	.592	.437	.325	.251
.75	.841	.702	.546	.419	.328
.80	.912	.805	.662	.528	.421
.85	.960	.893	.778	.650	.533
.90	.983	.956	.884	.781	.665
.95	.999	.992	.965	.910	.820
1.00	1.000	1.000	1.000	1.000	1.000

Table III

$$F_X(x) = \frac{\int_0^x y^A (1-y)^G \, dy}{\int_0^1 y^A (1-y)^G \, dy}$$

M is the position of the mode.

DISTRIBUTION FUNCTION OF THE STANDARD BETA FUNCTION WITH D = 7

X	A= .000 G=3.899 M= .000	A= .517 G=4.655 M= .100	A=1.329 G=5.316 M= .200	A=2.399 G=5.598 M= .300	A=3.569 G=5.354 M= .400	A=4.625 G=4.625 M= .500
.05	.189	.097	.026	.003	.000	.000
.10	.154	.128	.071	.022	.004	.000
.15	.530	.401	.225	.083	.020	.004
.20	.651	.540	.361	.175	.058	.014
.25	.745	.658	.497	.292	.124	.039
.30	.818	.754	.622	.423	.219	.085
.35	.874	.830	.729	.554	.336	.157
.40	.915	.886	.815	.675	.465	.254
.45	.944	.927	.881	.777	.595	.371
.50	.965	.956	.928	.857	.713	.500
.55	.979	.975	.959	.915	.812	.629
.60	.988	.987	.979	.954	.888	.746
.65	.994	.993	.990	.978	.940	.843
.70	.997	.997	.996	.991	.972	.915
.75	.999	.999	.999	.997	.989	.961
.80	1.000	1.000	1.000	.999	.997	.986
.85	1.000	1.000	1.000	1.000	.999	.996
.90	1.000	1.000	1.000	1.000	1.000	1.000
.95	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

X	A=5.354 G=3.569 M= .600	A=5.597 G=2.399 M= .700	A=5.316 G=1.329 M= .800	A=4.656 G= .517 M= .900	A=3.899 G= .000 M=1.000
.05	.000	.000	.000	.000	.000
.10	.000	.000	.000	.000	.000
.15	.001	.000	.000	.000	.000
.20	.003	.001	.000	.000	.000
.25	.011	.003	.001	.001	.001
.30	.028	.009	.004	.003	.003
.35	.060	.022	.010	.007	.006
.40	.112	.046	.021	.013	.011
.45	.188	.085	.041	.025	.020
.50	.287	.143	.072	.044	.034
.55	.405	.223	.119	.073	.053
.60	.535	.325	.185	.114	.082
.65	.664	.446	.271	.170	.121
.70	.781	.577	.378	.246	.174
.75	.876	.708	.503	.342	.244
.80	.942	.825	.639	.460	.335
.85	.980	.917	.775	.599	.451
.90	.996	.974	.894	.752	.597
.95	1.000	.997	.974	.903	.778
1.00	1.000	1.000	1.000	1.000	1.000

Table IV

$$F_{x^*}(x) = \frac{\int_0^x y^A (1-y)^G \, dy}{\int_0^1 y^A (1-y)^G \, dy}$$

M is the position of the mode

DISTRIBUTION FUNCTION OF THE STANDARD BETA FUNCTION WITH D = 8

X	A= .000 G=4.916 M= .000	A= .686 G=6.175 M= .100	A=1.825 G=7.300 M= .200	A=3.348 G=7.812 M= .300	A=5.010 G=7.515 M= .400	A=6.050 G=6.050 M= .500
.05	.223	.101	.018	.001	.000	.000
.10	.172	.145	.069	.013	.001	.000
.15	.593	.445	.223	.063	.009	.001
.20	.719	.598	.378	.154	.036	.007
.25	.803	.721	.534	.283	.095	.024
.30	.872	.815	.672	.434	.192	.062
.35	.918	.883	.784	.585	.321	.129
.40	.949	.929	.867	.718	.470	.228
.45	.969	.960	.924	.825	.620	.356
.50	.983	.979	.960	.901	.751	.500
.55	.991	.989	.981	.949	.854	.644
.60	.995	.995	.992	.977	.924	.772
.65	.998	.998	.997	.991	.966	.871
.70	.999	.999	.999	.997	.988	.938
.75	1.000	1.000	1.000	.999	.997	.976
.80	1.000	1.000	1.000	1.000	.999	.993
.85	1.000	1.000	1.000	1.000	1.000	.999
.90	1.000	1.000	1.000	1.000	1.000	1.000
.95	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

X	A=7.511 G=5.007 M= .600	A=7.811 G=3.348 M= .700	A=7.030 G=1.753 M= .800	A=6.175 G= .686 M= .900	A=4.916 G= .000 M=1.000
.05	.000	.000	.000	.000	.000
.10	.000	.000	.000	.000	.000
.15	.000	.000	.000	.000	.000
.20	.001	.000	.000	.000	.000
.25	.003	.001	.000	.000	.000
.30	.012	.003	.001	.001	.001
.35	.034	.009	.004	.002	.002
.40	.076	.023	.009	.005	.004
.45	.146	.051	.021	.011	.009
.50	.249	.099	.043	.021	.017
.55	.380	.175	.081	.040	.029
.60	.530	.282	.139	.071	.049
.65	.679	.415	.222	.117	.078
.70	.808	.566	.334	.185	.121
.75	.905	.717	.470	.279	.182
.80	.964	.846	.624	.402	.267
.85	.991	.937	.777	.555	.382
.90	.999	.985	.904	.728	.536
.95	1.000	.999	.981	.899	.738
1.00	1.000	1.000	1.000	1.000	1.000

Table V

$$F_{X}(x) = \frac{\int_0^x y^A \, (1-y)^G \, dy}{\int_0^1 y^A \, (1-y)^G \, dy}$$

M is the position of the mode.

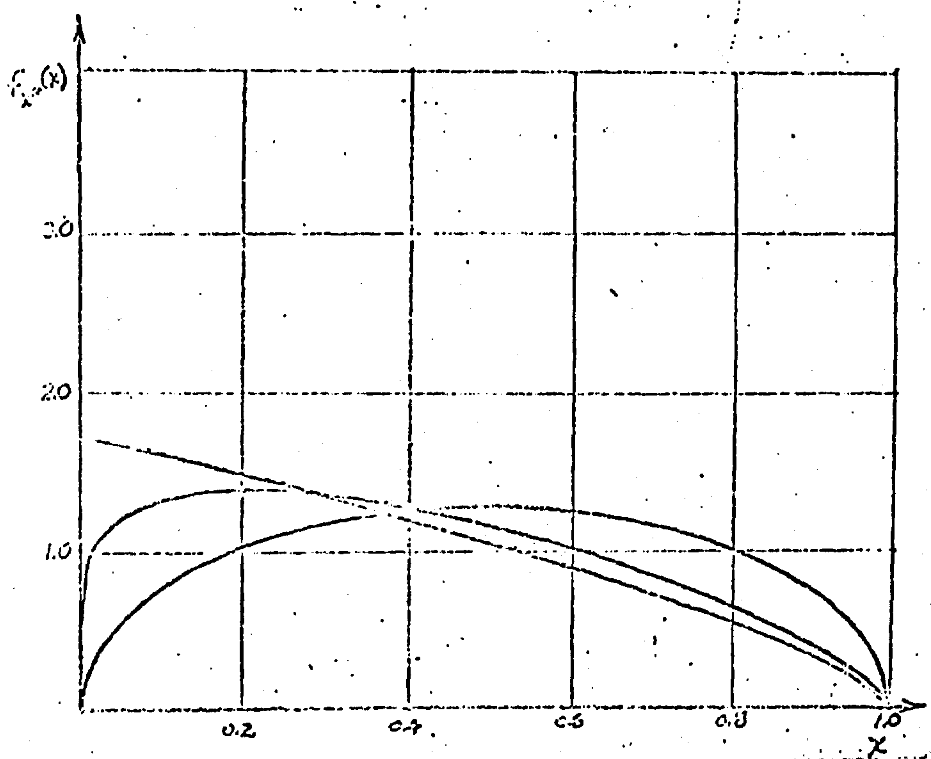


Figure 11

Beta Density Functions for d = 4

Modes at 0, .2, and .5

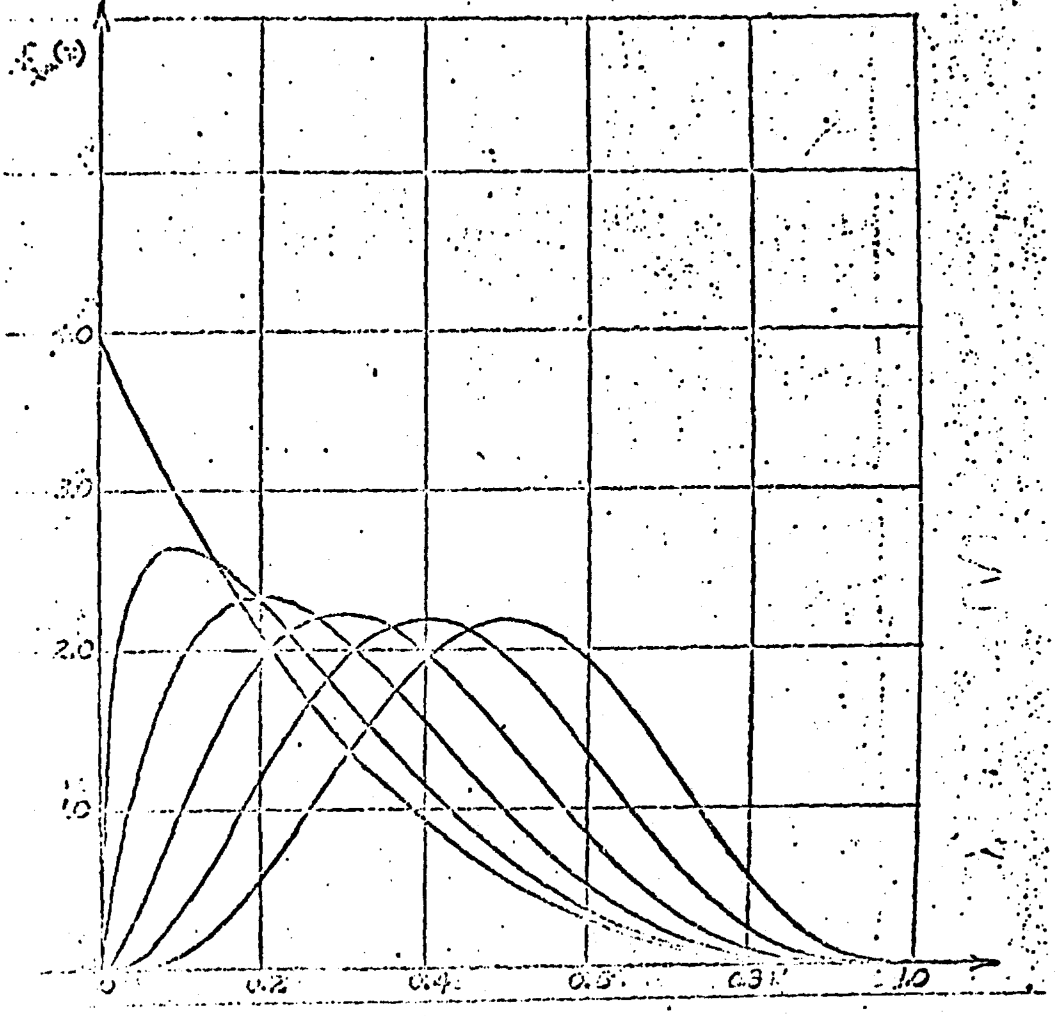


Figure 12

Beta Density Functions for d = 6

Modes at 0, .1, .2, .3, .4, and .5

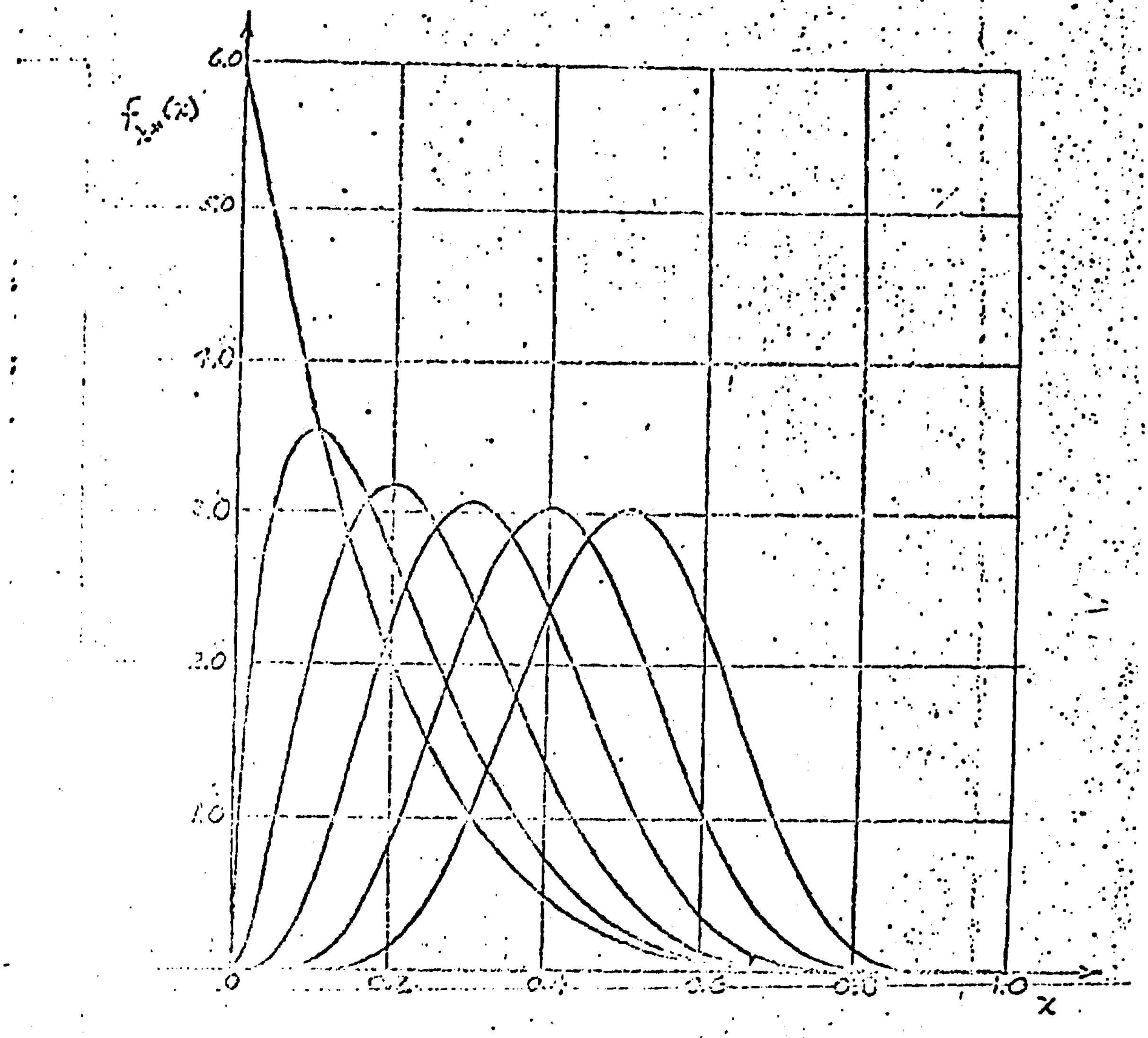


Figure 13

Beta Density Functions for d = 8

Modes at 0, .1, .2, .3, .4, and .5

APPENDIX II

A Simpler Method for Operating With the Beta Distribution

Since the Monte Carlo method does not require that the variance of t* be calculated and applied in order to evaluate P_D, we may consider dispensing with the requirement that the variance of t* be constant over the range of m. This requirement is not necessarily a natural one. The variance of the triangular distribution, for example, defined naturally by $\emptyset$, M, and P, is given by:

$$\sigma^2(t^*) = 2\left(\frac{P-\emptyset}{6}\right)^2 - \frac{(P-M)(M-\emptyset)}{18} \tag{18}$$

We see that the variance for this distribution varies from a maximum of $2\left(\frac{P-\emptyset}{6}\right)^2$ with m at the extremes of its range, to $\frac{3}{2}\left(\frac{P-\emptyset}{6}\right)^2$ with m at the midpoint. The variance of the SinCos function, (Appendix IV), another function naturally specified by the three estimates, behaves in the same general manner.

We may simplify the determination of the Beta function parameters, avoiding the necessity to solve the cubic equation, (10), by the following procedure:

Fix the variance of the symmetric Beta function, (m = .5), by letting

$$\sigma^2(t^*)\Big|_{M = \frac{P+\emptyset}{2}} = \left(\frac{P-\emptyset}{d}\right)^2 \tag{19}$$

where d is an arbitrary positive constant $\geq \sqrt{12}$, or

$$\sigma^2(x^*)\Big|_{m = .5} = \frac{1}{d^2} \tag{20}$$

For the case where m = .5, equations (8), (9), and (20) yield the following:

$$\alpha = \gamma = \frac{d^2 - 12}{8} \qquad (21)$$

Define $$C = \frac{d^2}{8} - 2 \qquad (22)$$

Then, for any value of m in its range, let α and γ be determined as follows:

$$\begin{aligned} \gamma &= C, \quad 0 \leq m \leq .5 \\ \alpha &= C, \quad .5 \leq m \leq 1 \end{aligned} \qquad (23)$$

The remaining parameter may be determined by equation (8). The variance of x*, as determined by equation (9), then varies with m as shown in Figure 11.

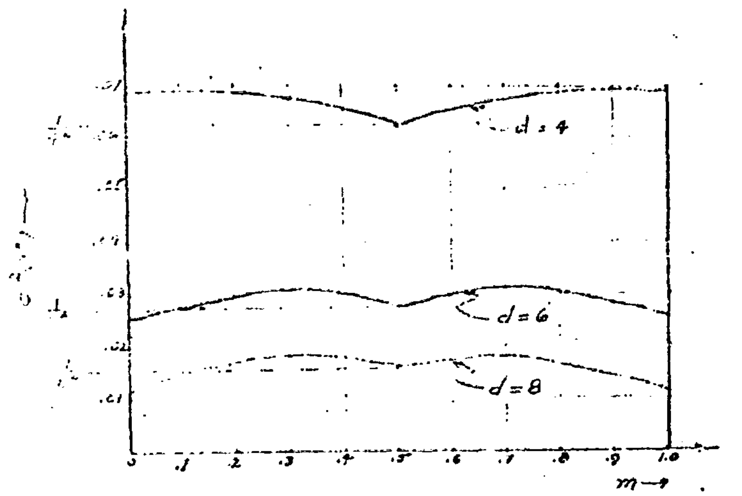


Figure 14

Change of Variance with Mode Position

The mean, E[x*], may be calculated directly by equation (12). The following close linear approximations for E[x*], for several values of d,

show the relative weights given to Ó, M, and P in the determination of expected value, hence give an intuitive feeling for the relationship between d and the amount of uncertainty with which m is located.

d	$E[x^*]$	
4	$\frac{2Ó + M + 2P}{5}$	(24)
6	$\frac{Ó + 4M + P}{6}$	
8	$\frac{Ó + 6M + P}{8}$	

By this relatively simple procedure, the parameters of the Beta function for any desired values of d may be determined for the purpose of the Monte Carlo calculations. The Probability distribution functions needed for the transformation from u^* to x^*, may be extracted from a table of the Incomplete Beta Function. [1]

APPENDIX III

Use of the Triangular Probability Density in PERT Monte Carlo Calculations

The triangular probability density function is naturally defined by the three time estimates, Ø, M, and P. Since the Distribution function of this density may be calculated by a simple formula, the transformation from a sample, u, drawn from a uniform (0,1) distribution, to a sample from the desired triangular density, may be accomplished by a single, simple calculation. When this procedure can be used, a considerable saving in computer running time is effected.

The technique is illustrated below for the random variable, y*, having the general triangular distribution with lower limit, Ø, upper limit, P, and mode, M.

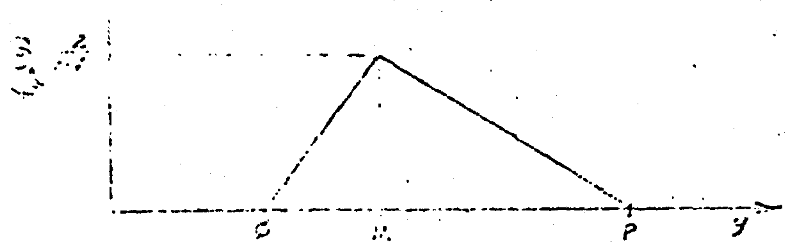


Figure 15

Triangular Density Function

The density function of y* is:

$$
\begin{aligned}
f_{y*}(y) &= \frac{2(y-\O)}{(P-\O)(M-\O)}, & \O \leq y \leq M \\
&= \frac{2(P-y)}{(P-\O)(P-M)}, & M \leq y \leq P \qquad (25) \\
&= 0, & \text{elsewhere}
\end{aligned}
$$

The distribution function, $F_{y*}(y) = \int_{-\infty}^{y} f_{y*}(z)dz$, is:

$$F_{y*}(y) = \frac{(y-O)^2}{(P-O)(M-O)}, \quad O \le y \le M$$

$$= \frac{M-O}{P-O} + \frac{M^2 - 2PM + 2Py - y^2}{(P-O)(P-M)}, \quad M \le y \le P \qquad (26)$$

$$= 0, \quad y \le O$$

$$= 1, \quad y \ge P$$

Now let $u = F_{y*}(y)$, and solve for the inverse function, $y = F_{y*}^{-1}(u)$, and we get:

$$y = O + \sqrt{u(P-O)(M-O)}, \quad 0 \le u \le \frac{M-O}{P-O} \qquad (27)$$

$$= P - \sqrt{(P-M)(P-O)(1-u)}, \quad \frac{M-O}{P-O} \le u \le 1.$$

If we draw a random number, u, from the uniform (0,1) distribution, and determine y by means of the function, (27), then y* has the desired triangular distribution.

The mean, $E[y*]$, is given exactly by:

$$E[y*] = \frac{O + M + P}{3} \qquad (28)$$

The variance, $\sigma^2(y*)$, is given by:

$$\sigma^2(y*) = 2\left(\frac{P-O}{6}\right)^2 - \frac{(P-M)(M-O)}{18} \qquad (29)$$

The term on the right varies from 0, when M is at either extreme, to $\frac{1}{2}\left(\frac{P-O}{6}\right)^2$, when M is at the midpoint.

APPENDIX IV

The SinCos Function

Another Probability Function Suitable for PERT Monte Carlo Calculations

Another probability function which is completely specified by its extremes and its mode is the following, which we shall call the SinCos function, for obvious reasons.

The probability density function is:

$$f_{sz}(z) = \frac{\pi}{2(P-O)} \sin \frac{\pi(z-O)}{2(M-O)} \; ; \quad O \le z \le M \tag{30}$$

$$= \frac{\pi}{2(P-O)} \cos \frac{\pi(z-M)}{2(P-M)} \; , \quad M \le z \le P$$

$$= 0, \qquad \text{elsewhere}$$

The function has the following shape, where M can take any position between O and P.

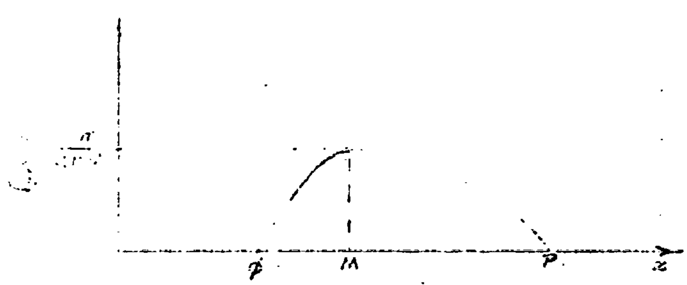


Figure 16

The SinCos Density Function

The distribution function, $F_{z*}(z) = \int_{-\infty}^{z} f_{z*}(t)\,dt$, is:

$$
\begin{aligned}
F_{z*}(z) &= 0, & z &\le \emptyset \\
&= \frac{M-\emptyset}{P-\emptyset}\left[1-\cos\frac{\pi(z-\emptyset)}{2(M-\emptyset)}\right], & \emptyset &\le z \le M \\
&= \frac{M-\emptyset}{P-\emptyset} + \frac{P-M}{P-\emptyset}\sin\frac{\pi(z-M)}{2(P-M)}, & M &\le z \le P \\
&= 1 & z &\ge P
\end{aligned}
\tag{31}
$$

Setting $u = F_{z*}(z)$, and solving for z, we obtain:

$$
\begin{aligned}
z &= \emptyset + \frac{2(M-\emptyset)}{\pi}\cos^{-1}\left[1-\frac{(P-\emptyset)u}{(M-\emptyset)}\right], & 0 &\le u \le \frac{M-\emptyset}{P-\emptyset} \\
&= M + \frac{2(P-M)}{\pi}\sin^{-1}\left[\frac{P-\emptyset}{P-M}\left(u-\frac{M-\emptyset}{P-\emptyset}\right)\right], & \frac{M-\emptyset}{P-\emptyset} &\le u \le 1
\end{aligned}
\tag{32}
$$

The above function transforms a random sample, u, from the uniform (0,1) distribution to the corresponding sample, z, from the SinCos distribution.

The mean, $E[z^*]$, of the SinCos distribution is:

$$E[z^*] = \frac{(\pi-2)\emptyset + (4-\pi)M + (\pi-2)P}{\pi} \tag{33}$$

Considered as a weighted average, we see that the extremes, $\emptyset$ and P, receive more weight than M.

The variance, $\sigma^2(z^*)$ is:

$$\sigma^2(z^*) = \frac{4(\pi-3)(P-\emptyset)^2 + (\pi^2-16\pi+40)(P-M)(M-\emptyset)}{\pi^2} \tag{34}$$

In a form for numerical comparison with the PERT Beta variance:

$$\sigma^2(z^*) = 2.06\left(\frac{P-\emptyset}{6}\right)^2 - .364\left(\frac{(P-M)(M-\emptyset)}{9}\right) \tag{35}$$

We see that $\sigma^2(x^*)$ is 1.68 + .15K times the PERT Beta variance, $\left(\frac{P-O}{6}\right)^2$, where $-1 \leq K \leq 1$, depending on the relative position of M.

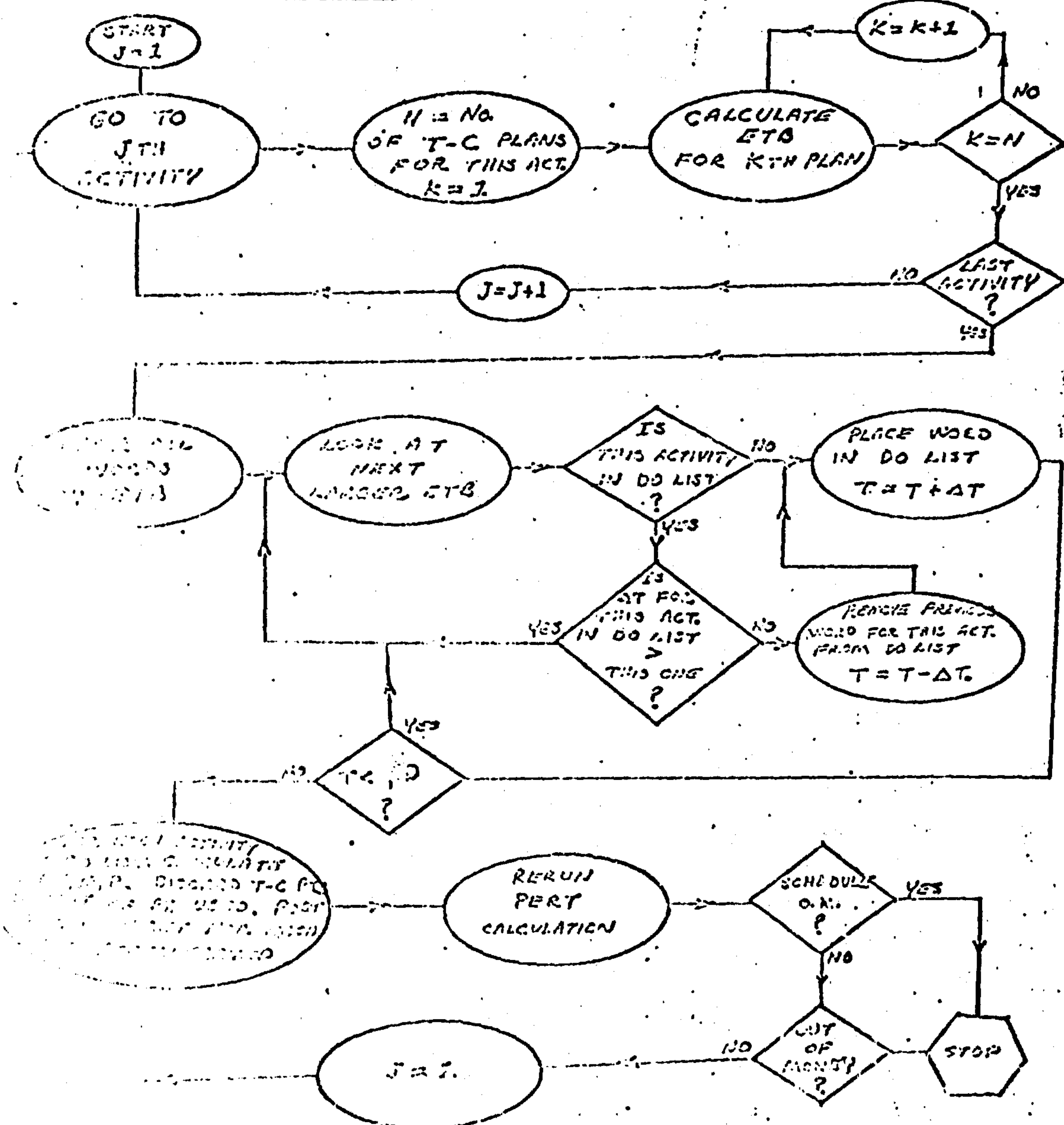
Although, with the SinCos function, the transformation from u* to x* can be accomplished by a single transformation equation, computer running time may be as great, or greater, than the table look-up routine, due to the number of operations required to calculate the $\sin^{-1}$ or the $\cos^{-1}$.

APPENDIX V

FLOW CHART FOR RESOURCE ALLOCATION USING CRITICALITY INDEX

For each T-C plan, form the following computer "word".

Activity No.	ETB	ΔT	ΔC



BIBLIOGRAPHY

1. Pearson, Karl, Tables of the Incomplete Beta Function, Biometrica, Cambridge, England, 1956.

2. Special Projects Office, Bureau of Naval Weapons, PERT Summary Report, Phase I, U. S. Government Printing Office, July 1958.

3. Parzen, E., Modern Probability Theory and its Applications, John Wiley & Sons, New York, 1960.

4. Clarke, R. W., An Introduction to Critical Path Analysis, Stanford University Thesis, March 1961.

5. International Business Machines Corporation, General Information Manual, PERT, E20-8067.

6. Clark, C. E., "The Pert Model for the Distribution of Activity Times," Operations Research 10, 405-406 (1962).

7. Cray, W. A., and E. M. Kidd, Critical Path Scheduling With Resource [illegible]ling on the IBM-7090, Union Carbide Nuclear Company, Oak Ridge, 1962.

8. Grubbs, F. E., "Attempts to Validate Certain PERT Statistics, or 'Picking on PERT'," Operations Research 10, 912-915 (1962).

9. [illegible]lian, G. N., PERT, New Management Planning and Control Technique, [illegible]ican Management Association, New York, 1962.

10. U. S. Department of Defense and NASA, PERT/COST Systems Design, U. S. Government Printing Office, June 1962.

11. Atomics International, PERT Manpower Program Description, AI, Canoga Park, California, 1963.

12. Callahan, V. F., Defense/Space Management Handbook, Callahan Publications, Washington, 1963.

13. [illegible]ner, H., "A General Network Approach to the Planning and Scheduling of a Research Project," Operations Research 11, 115-125 (1963).

14. [illegible]stone, J. S., "PERT in Research Management," Industrial Research, June 1963, 22-25.

15. [illegible] American Aviation Corporation, PERT, MK III Programmer's Manual, [illegible], Los Angeles, 1963.

16. [illegible], J. L., SIMPERT, A Monte Carlo Technique for Calculation of [illegible] Network, Atomics International, Canoga Park, California, [illegible]

17. Van Slyke, R. M., "Monte Carlo Methods and the PERT Problem," Operations Research 11, 839-860 (1963).

18. MacCrimmon, K. R. and C. A. Ryavec, "An Analytic Study of the PERT Assumptions," Operations Research 12, 16-37 (Jan. 1964).

19. Evarts, H. F., Introduction to PERT, Allyn and Bacon, Boston, 1964.

20. Control Data Corporation, Program Evaluation Review Technique, CDC, Minneapolis.